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Jets of line bundles on curves and Wronskians ☆

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ABSTRACT

We collect a few results about jets of line bundles on curves and Wronskians, with a special emphasis to those arising from the canonical involution of a hyperelliptic curve.

0. Introduction

This paper can be ideally divided into two parts. The former is concerned with some generalities on *jets of line bundles* on curves, whose well-known definition is quickly recalled in Section 2 following [14], while the latter is about *Wronskians* arising from double ramified coverings of the projective line (hyperelliptic curves). The two parts are obviously related, as Wronskians can be defined through jets. However, their interplay becomes more visible through the more general notion of *Wronskian of a section of a Grassmann bundle*, introduced and studied in the Dissertation [18], which is sketched in Section 5 with the hope of providing further details in a future paper.

The first part of this paper presents two main results: Theorems 3.3 and 4.3. The former claims that, for each $h \geq 0$, *a line bundle L on a smooth complex projective curve C of positive genus is generated by the global sections if and only if the same holds for J^hL, the jets of L of order h* (see 2.1) — a fact for which we did not find any reference in the literature. Equivalently, there is a morphism of the curve C to the Grassmannian G of the $(h+1)$-dimensional quotients of $H^0(J^hL)$. Any such morphism is highly degenerated, even for low positive values of h, in the sense that its image in the Plücker embedding of G is contained in a linear section of high codimension; see our Example 7.4. This fact, however, occurs with globally generated stable bundles as well, if their degree is not sufficiently low, as pointed out in [21].

Theorem 4.3, instead, shows that, for each $1 \leq i \leq h+1$, *there is an exact sequence of the form*

$$0 \longrightarrow J^{i-1}(L \otimes K^{\otimes h-i+1}) \longrightarrow J^hL \longrightarrow J^{h-i}L \longrightarrow 0, \tag{11}$$

which is well known for $i = 1$; see [14, p. 224]. Our proof of (11) is not direct. We prove the existence of an isomorphism between $J^{i-1}(L \otimes K^{\otimes h-i+1})$ and the kernel $N_{h,i}(L)$ of the canonical truncation $J^hL \to J^{h-i}L \to 0$, by showing that they both fit, as middle term, into a short exact sequence involving the same kernel and cokernel. As the two extensions are proportionals (Proposition 4.2) the two bundles are isomorphic because of standard facts on extensions; see e.g. [17, Lemma 3.3]. What did surprise us is that, if $i > 1$, the way to view $J^{i-1}(L \otimes K^{\otimes h-i+1})$ as a vector subbundle of J^hL is not canonical, because the latter has many automorphisms; see Remark 4.5.

We came to study the kernels $N_{h,i}(L)$ while we were looking at certain *Wronski maps* (Section 5). They associate *Wronskians* to *linear systems*; see e.g. [5,20] or, over the real numbers, [6,7]. Recall that a g^r_d on C is a pair (V, L), where $L \in Pic^d(C)$ and $V \in G(r+1, H^0(L))$ — the Grassmannian variety of $(r+1)$-dimensional vector subspaces of the global

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holomorphic sections of L. To each basis $\mathbf{v} = (v_i)$ of V, $0 \le i \le r$, one may attach a *Wronskian* $W(\mathbf{v})$: it is an element of a certain space $V_{g,d,r}$ of sections of a suitable line bundle whose dimension depends on d, g, r only; see Section 5.2. As changing the basis of V has the effect of multiplying $W(\mathbf{v})$ by a non-zero complex number, one gets a well-defined point $W_V := W(\mathbf{v}) \bmod \mathbb{C}^* \in \mathbb{P}V_{g,r,d}$, the *Wronskian* of V. Adopting the same terminology used in the literature when $C = \mathbb{P}^1$ and $L := O_{\mathbb{P}^1}(d)$ (see e.g. [6,7]), we call *Wronski map* the holomorphic function $V \mapsto W_V$. It is, in general, neither injective nor surjective as the following two extremal cases show. If $C = \mathbb{P}^1$, the Wronski map $G(r+1, H^0(O_{\mathbb{P}^1}(d))) \to \mathbb{P}V_{0,r,d}$ is a finite surjective morphism of degree equal to the Plücker degree of the Grassmannian $G(r+1, d+1)$: in particular it is not injective; see [5,20] and, over the real numbers, [6,7]. On the other hand, if C is hyperelliptic and $\mathcal{M} \in Pic^2(C)$ is the line bundle defining its unique g^1_2, then $G(2, H^0(\mathcal{M}))$ is just a point and the Wronski map to $\mathbb{P}V_{g,1,2}$ is trivially injective and not surjective.

The situation, however, gets more balanced once one updates the classical notion of *Wronskian of a linear system* to that of *Wronskian of sections of a Grassmann bundle*; see also [10]. More precisely, let $\rho_{r,d} : G(r+1, J^dL) \to C$ be the Grassmann bundle of $(r+1)$-dimensional subspaces of fibers of J^dL. The kernel bundles $N_{h,i}(L)$ provide a canonical subbundle filtration $N_{d,\bullet}(L)$ of J^dL which allows us to speak of $N_{d,\bullet}$-*Schubert subvarieties* in $G(r+1, J^dL)$. Among them a distinguished one lives, which in [18] is a baptized *Wronskian subvariety* of $G(r+1, J^dL)$. It is a Cartier divisor, being the zero scheme of a section $\mathbb{W}_r$ of a suitable line bundle over the total space of $\rho_{r,d}$. The *Wronskian* of a holomorphic section γ of $\rho_{r,d}$ is, by definition, $\gamma^*\mathbb{W}_r$; see Section 5. If $\mathcal{S}_r$ is the tautological bundle over $G(r+1, J^dL)$, let $\Gamma_{\mathtt{t}}(\rho_{r,d})$ be the space of holomorphic sections of $\rho_{r,d}$ such that $\rho^*_{r,d}\mathcal{S}_r$ is a trivial bundle. If $\gamma \in \Gamma_{\mathtt{t}}(\rho_{r,d})$, then $\gamma^*\mathbb{W}_r \in V_{d,g,r}$: we define a *Wronski map* $\Gamma_{\mathtt{t}}(\rho_{r,d}) \to \mathbb{P}V_{g,r,d}$ sending $\gamma \mapsto W_\gamma := \gamma^*\mathbb{W}_r \bmod \mathbb{C}^*$. The key remark is that *each* $g^r_d := (V, L)$ *on* C *defines a section* $\gamma_V \in \Gamma_{\mathtt{t}}(\rho_{r,d})$ and *its Wronskian* W_{γ_V} *coincides with the Wronskian* W_V *of the linear system* V.

If C is hyperelliptic and $\mathcal{M}$ is, as above, the bundle defining the hyperelliptic involution, then, by 6.1, $V_{g,1,2} = H^0(\mathcal{M}^{\otimes g+1})$, and Theorem 5.7 shows that *the Wronski map* $\Gamma_{\mathtt{t}}(\rho_{2,1}) \to \mathbb{P}H^0(\mathcal{M}^{\otimes g+1})$ *is dominant*. The interest of the proof lies in the fact that it requires a precise knowledge of the shape of a basis of $H^0(\mathcal{M}^{\otimes g+1})$. This is how we land into the second part of the paper, which is related with Ref. [4], and where we prove Theorem 6.6: *if* $\boldsymbol{\lambda} = (\lambda_0, \lambda_1)$ *is a basis of* $H^0(\mathcal{M})$, *then for each* $j \ge 0$ *the following direct sum decomposition holds:*

$$H^0(\mathcal{M}^{\otimes g+1+j}) = \mathrm{Sym}^j H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda}) \oplus \mathrm{Sym}^{g+1+j} H^0(\mathcal{M}). \tag{$*$}$$

In particular, if $j = 0$, the equality $(*)$ says that a basis of $H^0(\mathcal{M}^{\otimes g+1})$ is the union of the monomorphic image of a basis of $\mathrm{Sym}^{g+1}H^0(\mathcal{M})$ in $H^0(\mathcal{M}^{\otimes g+1})$ together with the Wronskian $W(\boldsymbol{\lambda})$. We have thus extended in an intrinsic way, and in the same spirit of [1, III, Section 3], the description of the pluricanonical systems to hyperelliptic curves; see 6.8. Further, the very shape of the decomposition $(*)$ shows that the isomorphic image of a hyperelliptic curve in $\mathbb{P}H^0(\mathcal{M}^{\otimes g+1+j})$ lies on a rational normal scroll $S(g+1+j, j)$ (Section 7.5). We know no reference where the decomposition $(*)$ is displayed in exactly the same way as we do. Our arguments are entirely intrinsic: we do not need to assume any Weierstrass equation for plane models of hyperelliptic curves which, on the contrary, is implied by the decomposition $(*)$. We show in fact that a hyperelliptic curve is *canonically*, up to a choice of the basis of $H^0(\mathcal{M})$, the zero locus of a weighted homogeneous polynomial and the Weierstrass equation turns out to be the affine equation of its restriction to an open part of the ambient weighted projective space; see 7.1.

1. Notation

Throughout this paper we shall work over the complex field $\mathbb{C}$.

1.1. Let $\pi : F \to X$ be a holomorphic vector bundle of rank n over a smooth projective variety X: it will be thought of as a holomorphically varying family of vector spaces, according to [13, p. 69]. If $P \in X$, the *fiber* of F at P is an n-dimensional complex vector space which will be denoted, throughout the paper, by F_P. Once F is identified with the locally free sheaf of its holomorphic sections, the fiber F_P is precisely the stalk of F at P modulo the maximal ideal m_P of the regular local ring $O_{X,P}$ of germs of holomorphic functions around P. If $f_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_n(\mathbb{C})$ are (holomorphic) transition functions for F with respect to a trivializing open covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of X, a *holomorphic section* $s = (s_\alpha)$ of F is a collection of holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}$ such that $s_\alpha(P) = f_{\alpha\beta}(P)s_\beta(P)$, for each $\alpha, \beta \in \mathcal{A}$ and each $P \in U_\alpha \cap U_\beta$. A holomorphic section is then a holomorphic function $s : X \to F$, defined by $s(P) = \phi_\alpha^{-1}(P, s_\alpha(P))$ where $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$ is a trivialization of F over $U_\alpha \ni P$. The definition does not depend on the open trivializing set containing P. The vector space of global holomorphic sections of F will be denoted by $H^0(F) := H^0(X, F)$. Notice that $s(P)$ denotes the value of s at the point $P \in X$, thought of as a point of the fiber F_P of F at P, and not the image s_P of s in the stalk of the *sheaf of sections of* F at P.

1.2. A holomorphic vector bundle F is said to be *generated by its global holomorphic sections* if the natural evaluation map $H^0(F) \mapsto F_P$ is onto for each $P \in X$.

1.3. If V is a (finite dimensional complex) vector space, $G(k, V)$ will denote the Grassmannian variety parameterizing k-dimensional vector subspaces of V while $G(V, k)$ that parameterizing k-dimensional quotients. Clearly $G(V, k) \cong G(n-k, V)$.

2. Jets of line bundles

2.1. Let C be a smooth projective curve of genus $g \ge 1$, $C \times C$ the fiber product over $Spec(\mathbb{C})$ and $p, q : C \times C \to C$ the projections onto the first and the second factor respectively. Let $\delta : C \to C \times C$ be the diagonal morphism and $\mathcal{I}$ the ideal sheaf of the diagonal in $C \times C$. The canonical bundle of C is $K := \delta^*(\mathcal{I}/\mathcal{I}^2)$. Given $L \in Pic^d(C)$, for each $h \geq 0$ one defines

$$J^hL := p_*\left(\frac{O_{C\times C}}{\mathcal{I}^{h+1}} \otimes q^*L\right) \tag{1}$$

the bundle of *jets* (or *principal parts*) of L of order h. As C is smooth, J^hL is a vector bundle on C of rank $h + 1$. If C were singular the right-hand side of (1) would not be locally free. But for locally complete intersection curves, one may use the locally free substitutes of the principal parts, constructed via a "paste and glue" procedure in [8,9], used in [12] and more elegantly described, in an intrinsic way, by Laksov and Thorup in [15].

2.2. By definition $J^0L = L$. Let us set, by convention, $J^{-1}L = 0$ – the vector bundle of rank 0. The fiber of J^hL over $P \in C$ will be denoted by J^h_PL – a complex vector space of dimension $h + 1$. The obvious exact sequence

$$0 \longrightarrow \frac{\mathcal{I}^{h-i+1}}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{C\times C}}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{C\times C}}{\mathcal{I}^{h-i+1}} \longrightarrow 0,$$

gives rise to an exact sequence

$$0 \longrightarrow N_{h,i}(L) \xrightarrow{\iota_{h,i}} J^hL \xrightarrow{t_{h,h-i}} J^{h-i}L \longrightarrow 0 \tag{2}$$

where $N_{h,i}(L) = p_*(\mathcal{I}^{h-i+1}/\mathcal{I}^{h+1} \otimes q^*L)$ is a vector bundle of rank i, the kernel of the natural truncation epimorphism $t_{h,h-i} : J^hL \to J^{h-i}L \to 0$. By definition one sees that $N_{h,0}(L) = 0$, and $N_{h,h+1}(L) = J^hL$. Furthermore [14, p. 224] $N_{h,1}(L) = p_*\left(\mathcal{I}^h/\mathcal{I}^{h+1} \otimes q^*L\right) = L \otimes K^{\otimes h}$, whence the well-known exact sequence

$$0 \longrightarrow L \otimes K^{\otimes h} \longrightarrow J^hL \longrightarrow J^{h-1}L \longrightarrow 0. \tag{3}$$

Proposition 2.3. *For each $h \geq 1$ and each $0 \leq j < i \leq h + 1$, there is a natural short exact sequence of vector bundles:*

$$0 \longrightarrow N_{h,j}(L) \longrightarrow N_{h,i}(L) \xrightarrow{n_{h,i,j}} N_{h-j,i-j}(L) \longrightarrow 0. \tag{4}$$

Proof. For each $0 \leq j < i$, $N_{h,j}(L)$ maps injectively to $N_{h,i}(L)$, by definition. In addition, let $n_{h,i,j}$ be the restriction of $t_{h,h-j}$ to $N_{h,i}(L)$. In fact $t_{h,h-j}(N_{h,i}(L)) \subseteq N_{h-j,i-j}(L)$. To see this notice that $\mathbf{u} \in N_{h,i}(L)$ is mapped to zero in $J^{h-i}L$, by definition. Thus the image of $\mathbf{u}$ through the truncation $J^hL \to J^{h-j}L$ is mapped to $0 \in J^{h-i}L$, i.e. belongs to $N_{h-j,i-j}(L)$, as desired. To show that $n_{h,i,j}$ is surjective, consider the following diagram with exact rows and commutative squares:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{h,i}(L) & \longrightarrow & J^hL & \longrightarrow & J^{h-i}L & \longrightarrow & 0 \\
\downarrow & & {\scriptstyle n_{h,i,j}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & N_{h-j,i-j}(L) & \longrightarrow & J^{h-j}L & \longrightarrow & J^{h-i}L & \longrightarrow & 0
\end{array}. \tag{5}$$

The first and third vertical arrows are epimorphisms while the fourth is a monomorphism. Hence $n_{h,i,j}$ is an epimorphism by the Five Lemma [16, p. 169]. □

Proposition 2.4. *Let $\deg(L) > 0$. For each $h \geq 1$ and each $0 \leq j \leq h$, $H^1(N_{h,j}(L)) = 0$.*

Proof. The proposition is trivially true if $j = 0$, as $N_{h,0}(L) = 0$. In addition, by Riemann–Roch, $h^1(N_{h,1}(L)) = h^1(L \otimes K^{\otimes h}) = 0$ (see 2.2) as we assumed that L has positive degree on a curve of positive genus. One now argues by induction on j. Assume that $h^1(N_{h,j}(L)) = 0$ for all $1 \leq j \leq i - 1 \leq h - 1$. Using (4) for $j = i - 1$, one has

$$0 \longrightarrow N_{h,i-1}(L) \longrightarrow N_{h,i}(L) \longrightarrow N_{h-i+1,1}(L) \longrightarrow 0$$

from which the long exact cohomology sequence

$$0 \to H^0(N_{h,i-1}(L)) \to H^0(N_{h,i}(L)) \to H^0(N_{h-i+1,1}(L)) \to 0 \to H^1(N_{h,i}(L)) \to 0$$

where we have used the inductive hypothesis and the fact that $h^1(N_{h-i+1,1}(L)) = h^1(L \otimes K^{\otimes h-i+1}) = 0$. Hence $h^1(N_{h,i}(L)) = 0$ as desired. □

2.5. Proposition 2.4 implies that the long exact cohomology sequence associated with (4) is

$$0 \longrightarrow H^0(N_{h,j}(L)) \longrightarrow H^0(N_{h,i}(L)) \longrightarrow H^0(N_{h-j,i-j}(L)) \longrightarrow 0, \tag{6}$$

for each $0 \leq j \leq i \leq h$. Furthermore, for $i = h + 1$ and $j = h$, (6) gives

$$0 \longrightarrow H^0(N_{h,h}(L)) \longrightarrow H^0(J^hL) \longrightarrow H^0(L) \longrightarrow 0. \tag{7}$$

2.6. For the sake of simplicity, assume now that L possesses global holomorphic sections, which is the case we shall deal with in the rest of this paper. The bundle J^hL can be constructed by gluing trivial pieces as follows. Let $\mathcal{U} := \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be an open covering of C, $z_\alpha : U_\alpha \to \mathbb{C}$ a local coordinate on U_α, $\alpha \in \mathcal{A}$, and $\ell_{\alpha\beta}(z_\beta) : U_\alpha \cap U_\beta \to \mathbb{C}^*$ transition functions of L

with respect to $\mathcal{U}$. If $\lambda = (\lambda_\alpha) \in H^0(L)$ (see 1.1) define on U_α the holomorphic function $\lambda_\alpha^{(i)}(z_\alpha) := d^i\lambda_\alpha/dz_\alpha^i$, for each $\alpha \in \mathcal{A}$ and each $1 \le i \le h$. Whenever $U_\alpha \cap U_\beta \neq \emptyset$, let $\ell_{\alpha\beta}^{(h)}(z_\beta) : U_\alpha \cap U_\beta \to Gl_{h+1}(\mathbb{C})$ be the square $(h+1) \times (h+1)$ matrix such that

$$(\lambda_\alpha(z_\alpha), \lambda_\alpha^{(1)}(z_\alpha), \dots, \lambda_\alpha^{(h)}(z_\alpha))^T = \ell_{\alpha\beta}^{(h)}(z_\beta) \cdot (\lambda_\beta(z_\beta), \lambda_\beta^{(1)}(z_\beta), \dots, \lambda_\beta^{(h)}(z_\beta))^T,$$

where ^{T} denotes transposition. It turns out that $\{\ell_{\alpha\beta}^{(h)}(z_\beta) \mid (\alpha, \beta) \in \mathcal{A} \times \mathcal{A}\}$ is a cocycle, which defines J^hL; see [8,9,11]. Thus

$$\mathcal{D}_h\lambda := (\lambda_\alpha(z_\alpha), \lambda_\alpha^{(1)}(z_\alpha), \dots, \lambda_\alpha^{(h)}(z_\alpha))^T,$$

is a global holomorphic section of J^hL, which, locally, looks like the local representation of λ together with its first h derivatives. We notice that for each $(\alpha, \beta) \in \mathcal{A} \times \mathcal{A}$, $\ell_{\alpha\beta}^{(h)}(z_\beta)$ is a lower triangular matrix. In addition, the (i, i) diagonal entry of the transition matrix, $0 \le i \le h$ is $\ell_{\alpha\beta}(z_\beta)\kappa_{\alpha\beta}(z_\beta)^i$, where $\kappa_{\alpha\beta}(z_\beta)$ are the transition functions of the canonical bundle. This is an alternative way to explain the sequence (3).

Furthermore, by definition of determinant bundle, whose transition functions are the determinants of the transition functions of $\ell_{\alpha\beta}^{(h)}(z_\beta)$, it turns out that

$$\bigwedge^{h+1} J^hL \cong L^{\otimes h+1} \otimes K^{\otimes \frac{h(h+1)}{2}}, \tag{8}$$

a formula that we shall need later on.

2.7. The operator $\mathcal{D}_h : H^0(L) \to H^0(J^hL)$ supplies a canonical splitting of the sequence (7). In fact, if

$$q_{h,j} : H^0(J^hL) \longrightarrow H^0(J^{h-j}L) \tag{9}$$

is the surjection induced by the truncations $J^hL \to J^{h-j}L$, then $q_{h,h}(\mathcal{D}_h\lambda) = \lambda$.

Definition 2.8. A section $\lambda \in H^0(C, L)$ vanishes at $P \in C$ with multiplicity at least $h + 1$ if $\mathcal{D}_h\lambda(P) = 0$.

Similarly, one says that $\lambda \in H^0(C, L)$ vanishes at P with multiplicity exactly $h + 1$ if $\mathcal{D}_h\lambda(P) = 0$ and $\mathcal{D}_{h+1}\lambda(P) \neq 0$, in which case one writes $ord_P\lambda = h + 1$.

Lemma 2.9. *Let $(\lambda_1, \dots, \lambda_j)$ be holomorphic sections of L, and $P \in C$ such that*

$$ord_P\lambda_i = n_i, \tag{10}$$

with $0 \le n_1 < \cdots < n_j$. Then $\lambda_1, \dots, \lambda_j$ are linearly independent.

Proof. By induction on the integer j. If $j = 1$ the proposition is trivially true: if $ord_P\lambda_1 = n$, then λ_1 is not zero. Suppose now that the property holds for all $(j-1)$-tuples of sections, and suppose that $\sum_{i=1}^j a_i\lambda_i = 0$ is any linear dependence relation between $\lambda_1, \dots, \lambda_j$, with $ord_P\lambda_1 < ord_P\lambda_2 < \cdots < ord_P\lambda_j$. Then

$$\mathcal{D}_{n_1}\left(\sum_{i=1}^j a_i\lambda_i\right)(P) = \sum_{i=1}^j a_i(\mathcal{D}_{n_1}\lambda_i)(P) = 0.$$

But $(\mathcal{D}_{n_1}\lambda_1)(P) \neq 0$, while $(\mathcal{D}_{n_1}\lambda_i)(P) = 0$ for all $2 \le i \le j$. Thus $a_1(\mathcal{D}_{n_1}\lambda_1)(P) = 0$, i.e. $a_1 = 0$ and one is left with a linear relation $\sum_{i=2}^j a_i\lambda_i = 0$ with $ord_P\lambda_2 < \cdots < ord_P\lambda_j$ implying $a_2 = \cdots = a_j = 0$, by induction. □

3. Jets of line bundles generated by the global sections

3.1. Let $q_{h,j}$ as in (2.7). For each $P \in C$ and each $\mu \in H^0(J^hL)$, one clearly has $q_{h,j}(\mu)(P) = t_{h,h-j}(\mu(P))$. If $\mu_i \in H^0(N_{h,i}(L)) \subseteq H^0(J^hL)$ then $q_{h,j}(\mu_i) = 0$ for all $j \ge i$, while by (6) for $j = i - 1$, $q_{h,i-1}(\mu_i) \in H^0(N_{h-i+1,1}(L))$. Because of the surjectivity of $q_{h,i-1}$, there are plenty of elements μ_i of $H^0(N_{h,i}(L))$ such that $q_{h,i-1}(\mu_i) \neq 0$ in $H^0(N_{h-i+1,1}(L))$.

Proposition 3.2. *Let $0 \neq \lambda \in H^0(L)$ and $\mu_i \in H^0(N_{h,i}(L))$ such that $q_{h,i-1}(\mu_i) \neq 0$, $1 \le i \le h$. Then*

$$(\mathcal{D}_h\lambda, \mu_1, \dots, \mu_h)$$

are linearly independent in $H^0(J^hL)$.

Proof. In fact, any linear dependence relation $a_0\mathcal{D}_h\lambda + a_1\mu_h + a_2\mu_{h-1} + \cdots + a_h\mu_1 = 0$ implies, for each $P \in C$, that

$$(a_0\mathcal{D}_h\lambda + a_1\mu_h + a_2\mu_{h-1} + \cdots + a_h\mu_1)(P) = a_0\mathcal{D}_h\lambda(P) + a_1\mu_h(P) + a_2\mu_{h-1}(P) + \cdots + a_h\mu_1(P) = 0$$

where the equalities are taken in J_P^hL. Then

$$0 = q_{h,h}(a_0\mathcal{D}_h\lambda + a_1\mu_h + \cdots + a_h\mu_1)(P) = a_0q_{h,h}(\mathcal{D}_h\lambda)(P) = a_0\lambda(P).$$

As $\lambda \neq 0$ there is at least a point P (and hence an open subset of C) such that $\lambda(P) \neq 0$, i.e. $a_0 = 0$. Suppose now that $a_j = 0$ for all $0 \le j \le i-1 \le h-1$. Then, applying $q_{h,h-i}$ to $a_i\mu_{h-i+1} + \cdots + a_h\mu_1 \in H^0(N_{h,h-i+1}(L))$, one has $a_iq_{h,h-i}(\mu_{h-i+1})(P) = 0$, for each $P \in C$. By hypothesis, there are points $P \in C$ such that $q_{h,h-i}(\mu_{h-i+1})(P) \neq 0$, i.e. $a_i = 0$. □

Theorem 3.3. *For each $h \geq 1$ and each $1 \leq i \leq h$, the vector bundle $N_{h,i}(L)$ is generated by its global sections; in addition J^hL is generated by its global sections if and only if L is.*

Proof. For each $P \in C$ one wants to find linearly independent sections of $N_{h,i}(L)$ such that their evaluation at P generate the fiber of $N_{h,i}(L)$ at P. First we use the fact that $H^0(N_{h,j}(L))$ can be seen as a subspace of sections of $N_{h,i}(L)$ for each $1 \leq j \leq i$, and then the fact that the map $H^0(N_{h,j}(L)) \to H^0(N_{h-j+1,1}) = H^0(L \otimes K^{\otimes h-j+1})$ is surjective $(1 \leq j \leq i)$. By applying Riemann–Roch formula one sees that $L \otimes K^{\otimes h-j+1}$ is generated by its global sections. Thus there exists $\xi_j \in H^0(L \otimes K^{\otimes h-j+1})$ such that $\xi_j(P) \neq 0$. Let $\mu_j \in H^0(N_{h,j}(L))$ such that $q_{h,j-1}(\mu_j)(P) = \xi_j(P)$. Then

$$\mu_1(P), \ldots, \mu_i(P)$$

span the fiber of $N_{h,i}(L)$ at P. In fact if $a_1\mu_1(P) + \cdots + a_i\mu_i(P) = 0$ is any linear dependence relation, one has

$$0 = t_{h,h-i+1}(a_1\mu_1 + \cdots + a_i\mu_i)(P) = a_i q_{h,i-1}(\mu_i)(P) = a_i\xi_i(P)$$

from which $a_i = 0$. Supposing that $a_k = 0$ for all $k \geq j+1$, one has

$$0 = t_{h,h-j+1}(a_1\mu_1 + \cdots + a_j\mu_j)(P) = a_j q_{h,j-1}(\mu_j)(P) = a_j\xi_j(P),$$

i.e. $a_j = 0$. This proves that all $N_{h,i}(L)$ are generated by the global sections for $1 \leq i \leq h$. The same argument works verbatim for $J^hL = N_{h,h+1}(L)$, but in this case one must add the hypothesis that L is generated by its global sections, in order to choose a $\xi_0 \in H^0(L)$ not vanishing at P. Conversely suppose that J^hL is generated by its global sections. Then there are $\mu_0, \mu_1, \ldots, \mu_h$ linearly independent holomorphic sections of J^hL such that

$$(\mu_0(P), \mu_1(P), \ldots, \mu_h(P))$$

generate J^h_PL. Then, there exists at least $0 \leq j \leq h$ such that $q_{h,h}(\mu_j)(P) = t_{h,0}(\mu_j(P)) \neq 0$, i.e. L is generated by its global sections. □

Corollary 3.4. *For each curve C and each $L \in Pic^d(C)$ which is generated by its global sections, there is a morphism $C \to G(H^0(J^hL), h+1)$, where $G(V, k)$ is as in 1.3.*

Proof. Via the standard map $P \mapsto \ker(\mathrm{ev}_P)$, where $\mathrm{ev}_P : H^0(J^hL) \longrightarrow J^h_PL$. □

In particular, if C is a curve of genus $g \geq 1$ and L is line bundle on C which is generated by its global sections, there is a morphism of C to $G(H^0(J^1L), 2)$ (see Example 7.4 below).

4. The truncation sequences

This section aims to show that the kernels $N_{h,i}(L)$ are themselves isomorphic, although not canonically, to jets of L twisted by powers of the canonical bundle.

Proposition 4.1. *Let $L \in Pic^d(C)$. Then for each $j \geq h \geq 2$ the group $\mathrm{Ext}^1(J^{h-2}L, L \otimes K^j)$ vanishes.*

Proof. If $h = 2$ the property is true, as $\mathrm{Ext}^1(L, L \otimes K^{\otimes j}) = H^1(K^{\otimes j})$ and, by Serre duality, $H^1(K^{\otimes j}) = 0$ for each $j \geq 2$. One now argues by induction on the integer $h \geq 2$. Assume that the property is true for each $2 \leq k \leq h-1$. Then, the exact sequence

$$0 \longrightarrow L \otimes K^{\otimes h-2} \longrightarrow J^{h-2}L \longrightarrow J^{h-3}L \longrightarrow 0$$

gives

$$\begin{aligned} 0 &\to Hom(J^{h-3}L, L \otimes K^j) \to Hom(J^{h-2}L, L \otimes K^j) \to Hom(L \otimes K^{\otimes h-2}, L \otimes K^j) \\ &\to \mathrm{Ext}^1(J^{h-3}L, L \otimes K^j) \to \mathrm{Ext}^1(J^{h-2}L, L \otimes K^j) \to \mathrm{Ext}^1(L \otimes K^{\otimes h-2}, L \otimes K^j) \to 0. \end{aligned}$$

If $j \geq h$, then $j \geq h-1$ and by induction $\mathrm{Ext}^1(J^{h-3}L, L \otimes K^j) = 0$. Furthermore

$$\mathrm{Ext}^1(L \otimes K^{\otimes h-2}, L \otimes K^j) = H^1(L^{-1} \otimes K^{\otimes 2-h} \otimes L \otimes K^j) = H^1(K^{j-h+2}) = 0.$$

Hence $\mathrm{Ext}^1(J^{h-2}L, L \otimes K^j) = 0$ as required. □

Proposition 4.2. *For each $L \in Pic^d(C)$ and each $h \geq 1$ one has $\mathrm{Ext}^1(J^{h-1}L, L \otimes K^{\otimes h}) = \mathbb{C}$.*

Proof. If $h = 1$ the property is true:

$$\mathrm{Ext}^1(J^0L, L \otimes K) = \mathrm{Ext}^1(L, L \otimes K) = H^1(L^{-1} \otimes L \otimes K) = H^1(K) = \mathbb{C}.$$

For $h \geq 2$, one applies the contravariant functor $Hom(\bullet, L \otimes K^{\otimes h})$ to the exact sequence

$$0 \to L \otimes K^{\otimes h-1} \to J^{h-1}L \to J^{h-2}L \to 0$$

obtaining

$$\begin{aligned} 0 &\to Hom(J^{h-2}L, L \otimes K^{\otimes h}) \to Hom(J^{h-1}L, L \otimes K^{\otimes h}) \to Hom(L \otimes K^{\otimes h-1}, L \otimes K^{\otimes h}) \\ &\to \mathrm{Ext}^1(J^{h-2}L, L \otimes K^{\otimes h}) \to \mathrm{Ext}^1(J^{h-1}L, L \otimes K^{\otimes h}) \to \mathrm{Ext}^1(L \otimes K^{\otimes h-1}, L \otimes K^{\otimes h}) \to 0. \end{aligned}$$

Now $\mathrm{Ext}^1(J^{h-2}L, L \otimes K^{\otimes h}) = 0$ by Proposition 4.1, while $\mathrm{Ext}^1(L \otimes K^{\otimes h-1}, L \otimes K^{\otimes h}) = H^1(K) = \mathbb{C}$. Hence $\mathrm{Ext}^1(J^{h-1}L, L \otimes K^{\otimes h}) \cong \mathbb{C}$, as claimed. □

Theorem 4.3. *The bundles $N_{h,i}(L)$ and $J^{i-1}(L\otimes K^{\otimes h-i+1})$ are isomorphic, for each $h\geq 1$ and $1\leq i\leq h+1$. Hence there exists a vector bundle morphism ψ making the sequence*

$$0\longrightarrow J^{i-1}(L\otimes K^{\otimes h-i+1})\xrightarrow{\psi} J^hL\longrightarrow J^{h-i}L\longrightarrow 0, \tag{11}$$

exact.

Proof. The property is trivially true for $h=1$ and $i=1$, as $N_{1,1}(L)=L\otimes K$. Assume that the proposition holds for all $1\leq k\leq h-1$ and all $1\leq i\leq k+1$. Induction on $1\leq i\leq h+1$. For $i=1$ one knows that $N_{h,1}(L)=L\otimes K^{\otimes h}$. The exact sequence (3) applied to the bundle $L\otimes K^{h-i+1}$ for $h=i-1$, gives

$$0\longrightarrow L\otimes K^{\otimes h}\longrightarrow J^{i-1}(L\otimes K^{\otimes h-i+1})\longrightarrow J^{i-2}(L\otimes K^{\otimes h-i+1})\longrightarrow 0.$$

On the other hand the bundle $N_{h,i}(L)$ fits into the exact sequence

$$0\longrightarrow L\otimes K^{\otimes h}\longrightarrow N_{h,i}(L)\longrightarrow N_{h-1,i-1}(L)\longrightarrow 0.$$

Using induction one gets an exact sequence

$$0\longrightarrow L\otimes K^{\otimes h}\longrightarrow N_{h,i}(L)\xrightarrow{\chi} J^{i-2}(L\otimes K^{\otimes h-i+1})\longrightarrow 0$$

where χ is the composition of $n_{h,i,1}$ (cf. (4)) with any isomorphism between $N_{h-1,i-1}(L)$ and $J^{i-2}(L\otimes K^{\otimes h-i+1})$. Thus both $N_{h,i}(L)$ and $J^{i-1}(L\otimes K^{\otimes h-i+1})$ are extensions of $J^{i-2}(L\otimes K^{\otimes h-i+1})$ by $L\otimes K^{\otimes h}$. Because of Proposition 4.2, applied to the bundle $L\otimes K^{\otimes h-i+1}$, one gets the equality $\mathrm{Ext}^1(J^{i-2}(L\otimes K^{\otimes h-i+1}), L\otimes K^{\otimes h})=\mathbb{C}$, and by [17, Lemma 3.3.], there exists a vector bundle isomorphism $\phi_i : N_{h,i}(L)\to J^{i-1}(L\otimes K^{\otimes h-i+1})$. The exact sequence (11), that generalizes the classical exact sequence (3), is then obtained from (2) by taking $\psi:=\iota_{h,i}\circ\phi_i$. □

Corollary 4.4. *There is a short exact sequence:*

$$0\longrightarrow H^0(J^{i-1}(L\otimes K^{\otimes h-i+1}))\longrightarrow H^0(J^hL)\longrightarrow H^0(J^{h-i}L)\longrightarrow 0.$$

Proof. It is the long exact cohomology sequence associated with (11). □

Remark 4.5. Theorem 4.3 says that $J^{i-1}(L\otimes K^{\otimes h-i+1})$ can be seen as a subbundle of J^hL, for each $1\leq i\leq h+1$. Such an identification, however, is not canonical unless $i=1$, in which case $N_{h,1}(L)=L\otimes K^{\otimes h}$, or $i=h+1$, in which case $N_{h,h+1}(L)=J^hL$. In fact if $\phi_i : N_{h,i}(L)\to J^{i-1}(L\otimes K^{\otimes h-i+1})$ is an isomorphism and τ is any non-trivial automorphism of $J^{i-1}(L\otimes K^{\otimes h-i+1})$, then $\tau\circ\phi_i : N_{h,i}(L)\to J^{i-1}(L\otimes K^{\otimes h-i+1})$ is an isomorphism as well and $\phi_i\neq\tau\circ\phi_i$. Now, J^hL has many non-trivial automorphism, for each L and each $h\geq 1$. To see this, first notice that each automorphism of J^1L induces an automorphism of J^hL for all $h\geq 2$, because of the epimorphism $J^hL\to J^1L\to 0$. Hence it suffices to show that J^1L possesses "many" non-trivial automorphisms. Let $\mathcal{R}$ be a trivializing cover of both L and K and let $\omega=(\omega_\alpha)\in H^0(K)$. If $v\in J^1_PL$, $P\in U_\alpha$, and $(u_{0,\alpha},u_{1,\alpha})^T$ is a vector of $\mathbb{C}^2$ representing v in the trivialization U_α, let

$$\psi_\alpha\begin{pmatrix}u_{0,\alpha}\\ u_{1,\alpha}\end{pmatrix}=\begin{pmatrix}u_{0,\alpha}\\ \omega_\alpha(P)u_{0,\alpha}+u_{1,\alpha}\end{pmatrix}.$$

A straightforward check shows that if $P\in U_\alpha\cap U_\beta$, then

$$\psi_\alpha(P)\circ\ell^{(1)}_{\alpha\beta}(z_\beta(P))=\ell^{(1)}_{\alpha\beta}(z_\beta(P))\psi_\beta(P),$$

i.e. the $\{\psi_\alpha\}$ glue together to give a global non-trivial isomorphism $\tau : J^1L\to J^1L$. □

5. Wronskians and Wronski maps

5.1. Linear systems. A g^r_d on a curve C of genus $g\geq 0$ is a pair (V,L), where $L\in Pic^d(C)$ and $V\in G(r+1,H^0(L))$. If C is a smooth curve of genus 1 and $P\in C$, then $h^0(O_C(2P))=2$, by Riemann–Roch. Then $O_C(2P)$ defines a g^1_2 on C, for each $P\in C$, making it into an *elliptic curve*. A *hyperelliptic curve* is a curve of genus $g\geq 2$ carrying a g^1_2, i.e. a line bundle $\mathcal{M}\in Pic^2(C)$ such that $h^0(\mathcal{M})=2$. By abuse of terminology, a *hyperelliptic curve* will be, in the following, any curve of genus $g\geq 1$ carrying a g^1_2.

5.2. Wronskians. Let (V,L) be a g^r_d. A point $P\in C$ is a *V-ramification point* if the map $C\times V\to J^rL$ mapping $(P,v)\mapsto\mathcal{D}_rv(P)$ drops rank at P. If $\boldsymbol{\lambda}:=(\lambda_0,\lambda_1,\ldots,\lambda_r)$ is a basis of $V\in G(r+1,H^0(L))$, the Wronskian of $\boldsymbol{\lambda}$ is the section

$$W(\boldsymbol{\lambda}):=\mathcal{D}_r\lambda_0\wedge\mathcal{D}_r\lambda_1\wedge\cdots\wedge\mathcal{D}_r\lambda_r\in H^0\left(\bigwedge^{r+1}J^rL\right)\cong H^0(L^{\otimes r+1}\otimes K^{\otimes\frac{r(r+1)}{2}}). \tag{12}$$

The *ramification scheme* of the given g^r_d is the zero scheme of $W(\boldsymbol{\lambda})$. After trivializations are taken, if $\lambda_i = (\lambda_{i,\alpha})$ (cf. 1.1), then $W(\boldsymbol{\lambda}) = (W_\alpha(\boldsymbol{\lambda}))$, where

$$W_\alpha(\boldsymbol{\lambda}) := \begin{vmatrix} \lambda_{0,\alpha} & \lambda_{1,\alpha} & \dots & \lambda_{r,\alpha} \\ \lambda'_{0,\alpha} & \lambda'_{1,\alpha} & \dots & \lambda'_{r,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{(r)}_{0,\alpha} & \lambda^{(r)}_{1,\alpha} & \dots & \lambda^{(r)}_{r,\alpha} \end{vmatrix}.$$

A *Weierstrass point* of C is a ramification point of the canonical linear system $(H^0(K), K)$. If C is hyperelliptic of genus $g \geq 2$ and P is a Weierstrass point, then $\mathcal{M} := O_C(2P)$ defines its unique g^1_2 [1].

5.3. Wronski maps. Changing the basis $\boldsymbol{\lambda}$ of V, the *Wronskian* $W(\boldsymbol{\lambda})$ gets multiplied by a non-zero constant: one then defines the *Wronskian of* V as $W_V := [W(\boldsymbol{\lambda})] \in \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$. The map

$$W : G(r+1, H^0(L)) \longrightarrow \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \tag{13}$$

defined by $V \mapsto W_V$ will be called *Wronski map* on $G(r+1, H^0(L))$. In the literature such a terminology usually refers to the special case when $C = \mathbb{P}^1$ and $L = L_d := O_{\mathbb{P}^1}(d)$. The Wronski map (13) is, in general, neither surjective nor injective. There are however two extremal cases: if $C = \mathbb{P}^1$, it is well known that the *Wronski map*

$$G\big(r+1, H^0(O_{\mathbb{P}^1}(d))\big) \to \mathbb{P}H^0\big(O_{\mathbb{P}^1}((r+1)(d-r))\big)$$

is a finite surjective morphism of degree equal to the Plücker degree of $G(r+1, d+1)$ [5]. In particular is not injective. On the other extreme, if $\mathcal{M} \in Pic^2(C)$ defines a g^1_2 on a hyperelliptic curve C, the Wronski map

$$G(2, H^0(\mathcal{M})) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K) \tag{14}$$

is trivially injective and not surjective, as $G(2, H^0(\mathcal{M}))$ is just a point! The situation becomes more uniform if one extends the notion of Wronskian to a wider class of objects, which are natural generalizations of linear systems, as follows.

5.4. Sections of Grassmann bundles. Let $L \in Pic^d(C)$ be a line bundle on a curve of genus $g \geq 0$ generated by the global sections and $\rho_{r,d} : G(r+1, J^dL) \to C$ be the Grassmann bundle of the $(r+1)$-dimensional subspaces of fibers of $\rho_{r,d}$. Let $0 \to \mathcal{S}_r \xrightarrow{\jmath_r} \rho^*_{r,d}J^dL \to \mathcal{Q}_r \to 0$ be the tautological sequence over the total space of $\rho_{r,d}$, where $\mathcal{S}_r$ stands for the universal subbundle of rank $r+1$ of J^dL. Letting i varying from 0 to $d+1$, the bundles $N_{d,i}(L)$ form a flag $N_{d,\bullet}$ of subbundles of J^dL:

$$0 = N_{d,0} \subset N_{d,1} \subset N_{d,2} \subset \cdots \subset N_{d,d} \subset N_{d,d+1} = J^dL.$$

For each $0 \leq i_0 < i_1 < \cdots < i_r \leq d$, define a Schubert (sub)variety of $G(r+1, J^dL)$:

$$\Omega_{(i_0,i_1,\dots,i_r)}(N_{d,\bullet}) = \{\Lambda \in G(r+1, J^dL) \mid \dim(\Lambda \cap \rho^*_{r,d}N_{d,d-i_j+1)}) \geq r+1-j\}.$$

Definition 5.5. The *Wronskian subvariety* of $G(r+1, J^dL)$ is:

$$\mathcal{W}(N_{d,\bullet}) := \Omega_{(01\dots,r-1,r+1)}(N_{d,\bullet}).$$

The *Wronskian variety* $\mathcal{W}(N_{d,\bullet})$ is a Cartier divisor. In fact, by its very definition, the degeneracy scheme of the natural map $\mathcal{S}_r \to \rho^*_{r,d}J^rL$ is obtained by composing the universal monomorphism $\jmath_r$ with the truncation $\rho^*_{r,d}t_{d,r}$. Thus $\mathcal{W}(N_{d,\bullet})$ is the zero scheme of the $N_{d,\bullet}$-*Wronskian* $\mathbb{W}_r := \bigwedge^{r+1}(\rho^*_{r,d}t_{d,r} \circ \jmath_r)$, a section of the line bundle $\bigwedge^{r+1}\rho^*_{r,d}J^rL \otimes (\bigwedge^{r+1}\mathcal{S}_r)^\vee$. Let

$$\Gamma(\rho_{r,d}) := \{\text{holomorphic maps } \gamma : C \to G(r+1, J^dL) \mid \rho_{r,d} \circ \gamma = id_C\}.$$

Definition 5.6. If $\gamma \in \Gamma(\rho_{r,d})$, the *Wronskian* $\mathbb{W}_\gamma$ of γ is:

$$\mathbb{W}_\gamma := \gamma^*\mathbb{W}_r \in H^0\left(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}} \otimes \bigwedge^{r+1}(\gamma^*\mathcal{S}_r)^\vee\right).$$

Let

$$\Gamma_{\mathtt{t}}(\rho_{r,d}) := \{\gamma \in \Gamma(\rho_{r,d}) \mid \gamma^*\mathcal{S}_r \text{ is a trivial subbundle of } J^dL\}.$$

We define the *Wronski map* on $\Gamma_{\mathtt{t}}(\rho_{r,d})$ as:

$$\begin{cases} \Gamma_{\mathtt{t}}(\rho_{r,d}) & \to & \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \\ \quad\gamma & \longmapsto & \gamma^*\mathbb{W}_r \mod \mathbb{C}^* \end{cases} \tag{15}$$

which extends the map (13). To see this, first notice that giving $\gamma \in \Gamma_{\mathtt{t}}(\rho_{r,d})$ amounts to specify a trivial subbundle $\gamma^*\mathcal{S}_r$ of J^dL, which itself determines an element $U_\gamma \in G(r+1, H^0(J^dL))$. Conversely, any $U \in G(r+1, H^0(J^dL))$ such that the natural evaluation map of section on points $C \times U \to J^dL$ is a bundle monomorphism, gives rise to a well-defined $\gamma_U \in \Gamma_{\mathtt{t}}(\rho_{r,d})$, by setting

$$\gamma_U(P) = \{u(P) \in J^d_PL \mid u \in U\} \in G(r+1, J^d_PL).$$

This shows that $\Gamma_{\mathtt{t}}(\rho_{r,d})$ can be indeed thought of as an open set of $G(r+1, H^0(J^dL))$. If $C = \mathbb{P}^1$ and $L = O_{\mathbb{P}^1}(d)$, one has indeed $\Gamma_{\mathtt{t}}(\rho_{r,d}) \cong G\big(r+1, H^0(J^dO_{\mathbb{P}^1}(d))\big) \cong G\big(r+1, H^0(O_{\mathbb{P}^1}(d))\big)$ and in this case $\Gamma_{\mathtt{t}}(\rho_{r,d})$ recovers all g^r_d on $\mathbb{P}^1$. In the general case, the injection $\mathcal{D}_d : H^0(L) \to H^0(J^dL)$ shows the inclusion of $G(r+1, H^0(L)) \subseteq \Gamma_{\mathtt{t}}(\rho_{r,d})$.

If $(\mathbf{u}_0, \ldots, \mathbf{u}_r)$ is a basis of $U \in G(r+1, H^0(J^dL))$ inducing a section γ_U of $\rho_{r,d}$, it is the matter of an easy exercise to show that $\gamma_U^*\mathbb{W}_r$ is the class modulo the action of $\mathbb{C}^*$ of the image of $\mathbf{u}_0 \wedge \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_r \in \bigwedge^{r+1} U$ in $H^0(\bigwedge^{r+1} J^rL)$. The "classical" Wronskian of $V \in G(r+1, H^0(L))$ is then precisely $\gamma^*_{\mathcal{D}_dV}\mathbb{W}_r$ which, if $(\lambda_0, \lambda_1, \ldots, \lambda_r)$ is a basis of V, gives back the expression (12).

Let C be hyperelliptic (in the sense of 5.1), $\mathcal{M}$ a bundle defining a g^1_2 and $\rho_{1,2} : G(2, J^1\mathcal{M}) \to C$ like in 5.4. Then, unlike the map (14), whose image is just a point:

Theorem 5.7. *The Wronski map $\Gamma_{\mathtt{t}}(\rho_{1,2}) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K)$ is dominant.*

The proof of 5.7 requires a precise description, which is interesting in its own, of the shape of the global sections of $\bigwedge^2 J^1\mathcal{M}$, which is the goal of the next section.

6. Pluricanonical systems on hyperelliptic curves

6.1. In the following C will be a complex hyperelliptic curve of genus $g \geq 1$ and $\mathcal{M}$ a bundle defining a g^1_2, unique if $g \geq 2$. In this case the canonical bundle is the $(g-1)$th power of $\mathcal{M}$: $K = \mathcal{M}^{\otimes g-1}$. This is obvious if $g = 1$, as K is trivial in this case; for $g \geq 1$, see [1, I-D9, p. 41]. Hence by (8), $\bigwedge^2 J^1\mathcal{M} = \mathcal{M}^{\otimes g+1}$. If $\boldsymbol{\lambda} = (\lambda_0, \lambda_1)$ is a basis of $H^0(\mathcal{M})$, then

$$\begin{cases} (\lambda_0 : \lambda_1) : C \longrightarrow \qquad \mathbb{P}^1 \\ \qquad\qquad P \longmapsto (\lambda_0(P) : \lambda_1(P)) \end{cases} \tag{16}$$

defines a ramified double cover of the projective line.

6.2. If $\lambda \in H^0(\mathcal{M})$, let $\mathcal{D}_1\lambda \in H^0(J^1\mathcal{M})$. There is a natural evaluation map

$$\bigwedge^2 H^0(J^1\mathcal{M}) \longrightarrow H^0\left(\bigwedge^2 J^1\mathcal{M}\right) = H^0(\mathcal{M}^{\otimes g+1}) \tag{17}$$

sending $\sigma_0 \wedge \sigma_1$ to the section of $\mathcal{M}^{\otimes g+1}$ defined by $P \mapsto \sigma_0(P) \wedge \sigma_1(P)$. The Wronskian $W(\boldsymbol{\lambda})$ of the basis $\boldsymbol{\lambda}$ of $H^0(\mathcal{M})$ is the image in $H^0(\bigwedge^2 J^1\mathcal{M})$ of $\mathcal{D}_1\lambda_0 \wedge \mathcal{D}_1\lambda_1$. Thus $W(\boldsymbol{\lambda})$ is a holomorphic section of $\mathcal{M}^{\otimes g+1}$, vanishing precisely at the $2g+2$ ramification points of the g^1_2, i.e. at the Weierstrass points of C. Since Riemann–Hurwitz's formula applied to a g^1_2 prescribes exactly $2g+2$ distinct ramification points, it follows that $W(\boldsymbol{\lambda})$ vanishes at each ramification point with multiplicity exactly 1; see also [1, I-E9].

6.3. Our next task is to revisit part of [4], in order to give an intrinsic description of the pluricanonical systems $H^0(K^{\otimes l})$ for hyperelliptic curves, essentially within the same spirit of [1, III, Section 3] to describe those for non-hyperelliptic ones. It is well known that non-hyperelliptic pluricanonical curves are projectively normal, meaning that the natural evaluation map $\mathrm{Sym}^lH^0(K) \to H^0(K^{\otimes l})$ is surjective for each $l \geq 0$. This is no longer true for hyperelliptic curves: for instance if $g = 2$, $Sym^lH^0(K) \to H^0(K^{\otimes l})$ fails to be surjective as soon as $l \geq 3$. Since $K = \mathcal{M}^{\otimes g-1}$, to compute a basis of $H^0(K^{\otimes l})$ for each $l \geq 0$, amounts to explicitly describe a basis of $H^0(\mathcal{M}^{\otimes a})$, for all $a \geq 0$. Notice that $h^0(\mathcal{M}^{\otimes a}) = a+1$ for all $0 \leq a \leq g-1$: in fact $\mathcal{M} = O_C(2P)$, where P is a Weierstrass point. Hence $h^0(O_C(2aP)) = 2+a-1$, as 2 is a Weierstrass non-gap at P; see e.g. [1, I-E, p. 41]. On the other hand $\mathcal{M}^{\otimes a}$ is non-special whenever $a \geq g$ and $h^0(\mathcal{M}^{\otimes a}) = 2a+1-g$. This can be easily seen by recalling that $\mathcal{M}$ is of the form $O_C(2P)$, where P is a ramification point of the g^1_2, and then applying Riemann–Roch formula. For each $a > 0$, let $\mathrm{Sym}^aH^0(\mathcal{M})$ be the ath symmetric power of the vector space $H^0(C, \mathcal{M})$. By convention set $\mathrm{Sym}^0H^0(\mathcal{M}) = \mathbb{C}$ and $\mathrm{Sym}^aH^0(\mathcal{M}) = 0$, if $a < 0$.

Proposition 6.4. *For each $a \in \mathbb{Z}$, the natural map $\mathrm{Sym}^aH^0(\mathcal{M}) \to H^0(\mathcal{M}^{\otimes a})$ is injective.*

Proof. This is trivially true for $a < 0$, as $\mathrm{Sym}^aH^0(\mathcal{M}) = 0$ and $\mathcal{M}^{\otimes a}$ has negative degree, implying $h^0(\mathcal{M}^{\otimes a}) = 0$. If $a = 0$ we have a non-zero homomorphism $\mathbb{C} \mapsto \mathbb{C}$ which is clearly an isomorphism. For $a \geq 1$, $\dim_{\mathbb{C}} \mathrm{Sym}^aH^0(\mathcal{M}) = a+1$ and if (λ_0, λ_1) is a basis of $H^0(\mathcal{M})$, then $\{\lambda_0^{a-i}\lambda_1^i\}_{0\leq i\leq a}$ generate the image of $\mathrm{Sym}^aH^0(\mathcal{M})$ in $H^0(\mathcal{M}^{\otimes a})$. It is then sufficient to show that they are linearly independent. One may assume, without loss of generality, that $\mathcal{D}_1\lambda_1(P) = 0$, for some $P \in C$, in which case P is a Weierstrass point of C and $2P$ is the Cartier divisor associated with the zero scheme of λ_1. Then $\lambda_0(P) \neq 0$, for if $\lambda_0(P) = 0$, the divisor associated with λ_0 would be $P + \iota(P)$ where ι is the hyperelliptic involution. As P is a ramification point of the g^1_2, $P = \iota(P)$, i.e. P is a double zero of λ_1 as well. Hence λ_0/λ_1 is a global holomorphic function on C, as λ_0 and λ_1 have both degree 2 and vanish along same divisor, i.e. it would be a constant, contradicting the fact that λ_0, λ_1 are linearly independent. Thus, for each $0 \leq i \leq a$, $\lambda_0^{a-i}\lambda_1^i$ vanishes at P with multiplicity $2i$ (Definition 2.8). Then $\{\lambda_0^{a-i}\lambda_1^i\}_{0\leq i\leq a}$ are linearly independent in $H^0(\mathcal{M}^{\otimes a})$ by Lemma 2.9. □

6.5. Because of Proposition 6.4, we shall identify $\mathrm{Sym}^a H^0(\mathcal{M})$ with its isomorphic image in $H^0(\mathcal{M}^{\otimes a})$. For example, we shall write $\mathrm{Sym}^a H^0(\mathcal{M}) \subset H^0(\mathcal{M}^{\otimes a})$. Now, for each $j \geq 0$, let $\mathrm{Sym}^j H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda})$ be the image of $\mathrm{Sym}^j H^0(\mathcal{M})$ in $H^0(\mathcal{M}^{\otimes g+1+j})$ through the multiplication-by-$W(\boldsymbol{\lambda})$ map $H^0(\mathcal{M}^{\otimes j}) \to H^0(\mathcal{M}^{\otimes g+1+j})$.

Theorem 6.6. *For each $a \in \mathbb{Z}$,*

$$H^0(\mathcal{M}^{\otimes a}) = \mathrm{Sym}^{a-g-1} H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda}) \oplus \mathrm{Sym}^a H^0(\mathcal{M}). \tag{18}$$

Proof. The sum on the right-hand side is clearly contained in the left-hand side. We already checked the proposition for $a \leq g$. If $a \geq g+1$, instead, $h^0(\mathcal{M}^{\otimes a}) = 2a + 1 - g$, again by Riemann–Roch. To show that the sum (18) is direct, it suffices to prove that the $2a + 1 - g$ elements

$$\{\lambda_0^{a-g-1-i}\lambda_1^i \cdot W(\boldsymbol{\lambda}), \lambda_0^{a-j}\lambda_1^j \mid 0 \leq i \leq a-g-1, 0 \leq j \leq a\} \tag{19}$$

are linearly independent in $H^0(\mathcal{M}^{\otimes a})$. Without loss of generality, one may assume that $\mathcal{D}_1\lambda_1(P) = 0$ for some Weierstrass point P. In this case $\lambda_0(P) \neq 0$ and P is a simple zero of the Wronskian $W(\boldsymbol{\lambda})$. It follows that

$$ord_P(\lambda_0^{a-g-1-i}\lambda_1^i \cdot W(\boldsymbol{\lambda})) = 2i + 1 \tag{20}$$

for each $0 \leq i \leq a - g - 1$, while

$$ord_P(\lambda_0^{a-j}\lambda_1^j) = 2j \tag{21}$$

for all $0 \leq j \leq a$. Such orders are all distincts (at the r.h.s. of (20) one has consecutive odd numbers while at the r.h.s. of (21) one has consecutive even numbers). The sections occurring in (19) are therefore linearly independent, again by Lemma 2.9. □

Corollary 6.7. *Let $\boldsymbol{\lambda} := (\lambda_0, \lambda_1)$ be a basis of $H^0(\mathcal{M})$. Then $(W(\boldsymbol{\lambda}), \lambda_0^{g+1}, \lambda_0^g\lambda_1, \ldots, \lambda_1^{g+1})$ is a $\mathbb{C}$-basis of $H^0(\mathcal{M}^{g+1})$.* □

Corollary 6.8. *The pluricanonical systems satisfy the equalities below:*

$$H^0(K) = \mathrm{Sym}^{g-1} H^0(\mathcal{M}), \qquad H^0(K^{\otimes 2}) = \mathrm{Sym}^2 H^0(K),$$
$$H^0(K^{\otimes l}) = \mathrm{Sym}^{(g-1)l} H^0(\mathcal{M}) \oplus \mathrm{Sym}^{(l-1)g-l-1} H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda}). \quad \square$$

6.9. The main reason why the morphism $C \to G(H^0(J^1L), 2)$ is too degenerate (cf. Example 7.4) is that the natural evaluation map (17) is not injective (see [21]). However, we may finally observe that

Theorem 6.10. *The map (17) is surjective.*

Proof. There is a canonical direct sum decomposition

$$H^0(J^1\mathcal{M}) = \mathcal{D}_1 H^0(\mathcal{M}) \oplus H^0(\mathcal{M} \otimes K) = \mathcal{D}_1 H^0(\mathcal{M}) \oplus H^0(\mathcal{M}^{\otimes g}).$$

Hence any section $\sigma \in H^0(J^1\mathcal{M})$ is of the form $\mathcal{D}_1\lambda + \mu$ where $\lambda \in H^0(\mathcal{M})$ and $\mu \in H^0(\mathcal{M}^{\otimes g}) = \mathrm{Sym}^g H^0(\mathcal{M})$. If $\sigma_i = \mathcal{D}_1\lambda_i + \mu_i$, $i = 0, 1$, then

$$\sigma_0 \wedge \sigma_1 = (\mathcal{D}_1\lambda_0 + \mu_0) \wedge (\mathcal{D}_1\lambda_1 + \mu_1) = \mathcal{D}_1\lambda_0 \wedge \mathcal{D}_1\lambda_1 + (\lambda_0\mu_1 - \lambda_1\mu_0).$$

Each element of $\tau \in H^0(\bigwedge^2 J^1\mathcal{M}) = H^0(\mathcal{M}^{\otimes g+1})$ is a unique linear combination

$$\tau = \sum_{i=0}^{g+1} a_i\lambda_0^{g+1-i}\lambda_1^i + a_{g+2}\mathcal{D}_1\lambda_0 \wedge \mathcal{D}_1\lambda_1.$$

If $a_{g+2} \neq 0$ one may write

$$\begin{aligned}
\tau &= \lambda_0 \sum_{i=0}^{g} a_i\lambda_0^{g-i}\lambda_1^i + a_{g+1}\lambda_1^{g+1} + a_{g+2}\mathcal{D}_1\lambda_0 \wedge \mathcal{D}_1\lambda_1 \\
&= a_{g+2}\left(\lambda_0 \sum_{i=0}^{g} \frac{a_i}{a_{g+2}}\lambda_0^{g-i}\lambda_1^i - \frac{a_{g+1}}{a_{g+2}}\lambda_1(-\lambda_1^g) + \mathcal{D}_1\lambda_0 \wedge \mathcal{D}_1\lambda_1\right) \\
&= a_{g+2}\left(\mathcal{D}_1\lambda_0 - \frac{a_{g+1}}{a_{g+2}}\lambda_1^g\right) \wedge \left(\mathcal{D}_1\lambda_1 + \sum_{i=0}^{g} \frac{a_i}{a_{g+2}}\lambda_0^{g-i}\lambda_1^i\right)
\end{aligned}$$

which proves that τ has a pre-image whenever $a_{g+2} \neq 0$. If $a_{g+2} = 0$, instead, one can write

$$\tau = \lambda_0 \sum_{i=0}^{g} a_i\lambda_0^{g-i}\lambda_1^i + a_{g+1}\lambda_1^{g+1} = \mathcal{D}_1\lambda_0 \wedge \sum_{i=0}^{g} a_i\lambda_0^{g-i}\lambda_1^i + \mathcal{D}_1\lambda_1 \wedge a_{g+1}\lambda_1^g$$

and the surjectivity is proven. □

6.11. Proof of Theorem 5.7. Recall that $\Gamma_{\mathrm{t}}(\rho_{1,2})$ is an open set of $G(2, H^0(J^1\mathcal{M}))$, coinciding with the subvariety of decomposable tensors of $\bigwedge^2 H^0(J^1L)$. Then, by the proof of 6.10, the Wronski map $\Gamma_{\mathrm{t}}(\rho_{1,2}) \subset G(2, H^0(J^1L)) \to \mathbb{P}H^0(\mathcal{M}^{\otimes g+1})$ surjects onto the open set of $H^0(\mathcal{M}^{\otimes g+1})$ defined by $a_{g+2} \neq 0$. Hence it is dominant. □

7. The Weierstrass equation for hyperelliptic curves

7.1. (Hyper)elliptic curves satisfy Weierstrass equations. This is well known, of course. Here is, however, another (very intrinsic) way to look at the Weierstrass equation for a complex hyperelliptic curve C of genus $g \geq 1$. Let $\mathcal{M} \in Pic^2(C)$ such that $h^0(\mathcal{M}) = 2$, and let $\boldsymbol{\lambda} = (\lambda_0, \lambda_1)$ be a basis of $H^0(\mathcal{M})$ defining a g^1_2, $C \to \mathbb{P}^1$ as in (16): $P \mapsto (\lambda_0(P), \lambda_1(P))$. Let $P_1, P_2, \dots, P_{2g+2}$ be the $2g+2$ Weierstrass points of C (or ramification points of some g^1_2 if $g = 1$). For each $1 \leq i \leq 2g+2$, let $(a_i : b_i) = (\lambda_0(P_i) : \lambda_1(P_i))$. Then $\lambda_i = a_i\lambda_1 - b_i\lambda_0$, $1 \leq i \leq 2g+2$, is such that $\mathcal{D}_1\lambda_i(P_i) = 0$. In fact, each λ_i vanishes at least once at P_i, and hence vanishes twice, because P_i is a Weierstrass point. Let

$$f(\boldsymbol{\lambda}) = \prod_{i=1}^{2g+2} \lambda_i = \prod_{i=1}^{2g+2} (a_i\lambda_1 - b_i\lambda_0).$$

Thus $f(\boldsymbol{\lambda})$ is a holomorphic section of the degree $4g+4$ line bundle $\mathcal{M}^{\otimes 2g+2}$, that vanishes at each P_i $(1 \leq i \leq 2g+2)$ with multiplicity 2. On the other hand the Wronskian $W(\boldsymbol{\lambda})$ vanishes at each P_i with multiplicity 1 (cf. 6.2), and hence, up to a constant, the equality $W(\boldsymbol{\lambda})^2 = f(\boldsymbol{\lambda})$ holds. If one sets $z = W(\boldsymbol{\lambda})$, $x_0 = \lambda_0$ and $x_1 = \lambda_1$, the hyperelliptic curve C can be hence seen, in a natural way, as the zero locus of a homogeneous polynomial

$$z^2 = f(x_0, x_1) \tag{22}$$

in the weighted projective space $\mathbb{P}(2g+2, 2, 2) = \mathbb{P}(g+1, 1, 1)$ (see [2,19]) with weighted homogeneous coordinates $(z : x_0 : x_1)$. If one sets $x = x_1/x_0$ and $y = z/x_0^{g+1}$ then

$$y^2 = (x - w_2)\dots(x - w_{2g+2}) \tag{23}$$

is precisely, up to a non-zero constant multiplying y^2, the *Weierstrass equation* of the hyperelliptic curve, where w_i is the affine coordinate $\lambda_1(P_i)/\lambda_0(P_i)$ of the image of the Weierstrass point P_i in the affine open set of $\mathbb{P}^1$ defined by $x_0 \neq 0$. So, indeed, we are distinguishing the affine non-homogeneous Weierstrass equation (23) from (22), its weighted projective homogeneous version (see also [19]).

Example 7.2. Let $(X_0 : X_1 : \dots : X_{g+2})$ be homogeneous coordinates of $\mathbb{P}^{g+2}$. The map

$$(W(\boldsymbol{\lambda}), \lambda_0^{g+1}, \lambda_0^g\lambda_1, \dots, \lambda_1^{g+1}) : C \to \mathbb{P}^{g+2}$$

is an embedding. It is apparent that the projective image of C is contained in the cone $S(g+1, 0)$, the scheme theoretical intersection of the $\binom{g+1}{2}$ quadric hypersurfaces in $\mathbb{P}^{g+2}$ defined by the equation

$$\mathrm{rk}\begin{pmatrix} X_1 & X_2 & \dots & X_{g+1} \\ X_2 & X_3 & \dots & X_{g+2} \end{pmatrix} \leq 1.$$

As in [4], one sees that the Weierstrass points of C $(W(\boldsymbol{\lambda}) = 0)$ lie on the rational normal curve obtained by intersecting $S(0, g+1)$ with $X_0 = 0$. The curve C is a quadric section of $S(0, g+1)$, deduced by the equation $W(\boldsymbol{\lambda})^2 = f(\boldsymbol{\lambda})$, with

$$f(\boldsymbol{\lambda}) = \prod_{i=1}^{2g+2} (a_i\lambda_1 - b_i\lambda_0) = \sum_{j=0}^{2g+2} A_j(\mathbf{a}, \mathbf{b})\lambda_0^{2g+2-j}\lambda_1^j$$

where $(a_i : b_i)$ are defined as in 7.1 and $A_j(\mathbf{a}, \mathbf{b})$ are the coefficients obtained by expanding the product above. The equation of a quadric cutting the curve on $S(0, g+1)$ is then

$$X_0^2 - \sum_{i=0}^{g+1} A_i X_1 X_{i+1} + \sum_{j=0}^{g} A_{g+2+j} X_{j+2} X_{g+2} = 0.$$

Remark 7.3. If $g = 2$, the projective image of C in $\mathbb{P}^4$ is then the scheme theoretical intersection of 4 quadrics, and in fact satisfies the formula displayed in [3, Corollary 2.5], for $g = 2$ and $r = 4$.

Example 7.4. The morphism $C \to G(H^0(J^1L), 2)$ as in Corollary 3.4, can be explicitly written as follows. Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_\alpha)$ be a basis of $H^0(\mathcal{M})$. If $P \in U_\alpha$ define

$$P \mapsto \begin{pmatrix} \lambda_{0,\alpha}(P) & \lambda_{1,\alpha}(P) & 0 & 0 & \dots & 0 \\ \lambda'_{0,\alpha}(P) & \lambda'_{1,\alpha}(P) & \lambda^g_{0,\alpha}(P) & \lambda^{g-1}_{0,\alpha}\lambda_{1,\alpha}(P) & \dots & \lambda^g_{1,\alpha}(P) \end{pmatrix} \quad \mathrm{mod}\, Gl_2(\mathbb{C}).$$

It is apparent from the explicit map above, that the image of the curve in the Plücker embedding of the Grassmannian is very degenerate, being contained in many hyperplanes. The curve is in fact contained in a $g+2$-dimensional linear subvariety of $\mathbb{P}^{\binom{g+3}{2}-1}$. The intersection of the pfaffians of the skew symmetric matrices defining the ideal of the Grassmannian $G(H^0(J^1\mathcal{M}), 2)$ in its Plücker embedding, cuts precisely the cone $S(0, g+1)$. Hence the embedding in $G(H^0(J^1L), 2)$ factorizes through that in $\mathbb{P}^{g+2}$, described in 7.2.

7.5. More generally, consider $\mathbb{P}^{g+2+2a}$ with homogeneous coordinates

$$(X_0 : X_1 : \ldots : X_a : X_{a+1} : \ldots : X_{2a+g+2}).$$

Let $\phi : C \to \mathbb{P}^{g+2a+2}$ be the embedding defined by the equations

$$\begin{cases} X_i &= \lambda_0^{a-i}\lambda_1^i \cdot W(\boldsymbol{\lambda}) \quad \text{for} \quad 0 \le i \le a \\ X_{a+1+j} &= \lambda_0^{g+1+a-j}\lambda_1^j \quad \text{for} \quad 0 \le j \le g+a+1. \end{cases}$$

The equations of the embedding clearly show that the image of C in $\mathbb{P}^{g+2+2a}$ is a curve of degree $2(g+1+a)$ lying on a rational normal scroll $S(a, g+1+a)$, a surface of degree $2a+g+1$. The ideal of the curve in $\mathbb{P}^{g+2+2a}$ is obtained by eliminating λ_0 and λ_1 from the ideal

$$J := (X_i - \lambda_0^{a-i}\lambda_1^i \cdot W(\boldsymbol{\lambda}), \qquad X_{a+1+j} - \lambda_0^{g+1+a-j}\lambda_1^j, \qquad W(\boldsymbol{\lambda})^2 - f(\boldsymbol{\lambda}))_{\substack{0 \le i \le a \\ 0 \le j \le g+1+a}}$$

Indeed, the ideal of the curve is minimally generated by quadrics. This is shown in [4].

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